

CU-TP-989, ITP-99/32

HIGGS BUNDLES AND FOUR MANIFOLDS

JAE-SUK PARK

Department of Physics, Columbia University
 538 West 120th Street, New York, N.Y. 10027, U.S.A¹
 (jspark@phys.columbia.edu)

and

Institute for Theoretical Physics, University of Amsterdam
 Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Abstract

It is known that the Seiberg-Witten invariants, derived from supersymmetric Yang-Mill theories in four-dimensions, do not distinguish smooth structure of certain non-simply-connected four manifolds. We propose generalizations of Donaldson-Witten and Vafa-Witten theories on a Kähler manifold based on Higgs Bundles. We showed, in particular, that the partition function of our generalized Vafa-Witten theory can be written as the sum of contributions our generalized Donaldson-Witten invariants and generalized Seiberg-Witten invariants. The resulting generalized Seiberg-Witten invariants might have, conjecturally, information on smooth structure beyond the original Seiberg-Witten invariants for non-simply-connected case.

1 Introduction

It is more than a decade ago that Witten introduced a quantum field theoretic formulation [1] of the four-dimensional differential topological invariants of Donaldson [2][3]. In this approach Donaldson invariants are defined as certain correlation functions of twisted $N = 2$ spacetime supersymmetric Yang-Mills (SYM) theory in four dimensions. This approach eventually opened up a new horizon in mathematics via the quantum properties of underlying physical theory, which is uncovered by Seiberg and Witten [4][5]. The resulting Seiberg-Witten invariants are much more simple, while carrying the same information for the smooth structure of four manifolds as the Donaldson-Witten invariants [6]. The Donaldson-Witten theory can be

¹The present address

generalized by twisting more general $N = 2$ SYM theory with hypermultiplets [7][8]. Such a theory can be also solved by the physical solution of the underlying $N = 2$ SYM theory [9][10][11][12][13]. However, those invariants defined by such a theory carry the same information as Seiberg-Witten invariants for the smooth structure of four manifolds. It is also shown that Gromov-Witten invariants are equivalent to Seiberg-Witten invariants [14].

On the other hand, it has been demonstrated that the Seiberg-Witten invariant alone does not distinguish smooth structures of certain, at least non-simply-connected, four manifolds belong to a same homeomorphism class [15][16]. Thus it is a challenging open problem to define four manifold invariants beyond Donaldson-Witten or Seiberg-Witten invariants.

The purpose of this paper is to propose candidates of such invariants restricting to the Kähler cases. We will take quantum field theoretic approach to these invariants. Our generalization of Donaldson-Witten theory involves the moduli space of stable Higgs bundles of Simpson [17][18][19] instead of the moduli space of stable bundles. We also propose the similar generalization of Vafa-Witten theory [20] - a twisted $N = 4$ SYM theory [21], as well as related generalization of Seiberg-Witten theory. We conjecture that the equivalence between Donaldson-Witten and Seiberg-Witten invariants remains valid to their generalized versions. We follow a general approach to constructing cohomological field theory with a Kähler structure, developed in [22]. Compared with Donaldson-Witten or Vafa-Witten theories our models do not have underlying spacetime supersymmetric theory. Our models, nevertheless, are "connected" to the physical $N = 2$ or $N = 4$ SYM theories by certain renormalization group flows, and those physical theories reside in particular of fixed points. Our conjecture is that the other fixed points give rise to new invariants of four manifold beyond Donaldson-Witten and Seiberg-Witten invariants.

2 Generalized Donaldson-Witten Theory

Donaldson-Witten theory (twisted $N = 2$ SYM) on a Kähler surface [23][24][25] is an example of $N_c = (2, 0)$ supersymmetric gauged sigma model in zero-dimensions [22]. Such a model is classified by a Kähler target space \mathcal{A} with a group \mathcal{G} acting as an isometry, which determines a \mathcal{G} -equivariant momentum map $\mu : \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^*$. We further have a Hermitian holomorphic vector bundle $\mathbb{E} \rightarrow \mathcal{A}$ with \mathcal{G} -equivariant holomorphic section \mathfrak{S} . Then the bosonic part of the path integral reduces to, provided that we are evaluating correlation functions for supersymmetric observables, an integration

over $\mathcal{M} := \mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta)/\mathcal{G}$; the solution space of the following equations, modulo \mathcal{G} ,

$$\begin{aligned}\mathfrak{S} &= 0, \\ \mu - \zeta &= 0,\end{aligned}\tag{2.1}$$

where ζ is the Fayet-Iliopoulos (FI) term. Those observables correspond to elements of \mathcal{G} -equivariant cohomology of \mathcal{A} . The correlation functions of such observables are identified with intersection numbers of homology cycles, represented by the observables, in $[e(\mathbb{V})]$, where $[e(\mathbb{V})]$ denotes the cycle in \mathcal{M} Poincaré dual to the Euler class $e(\mathbb{V})$ of the anti-ghost bundle \mathbb{V} over \mathcal{M} . If the model has actually $N_c = (2, 2)$ supersymmetry the anti-ghost bundle \mathbb{V} can be identified with the tangent bundle $T\mathcal{M}$ and the partition function is the Euler characteristic of \mathcal{M} . The moral underlying cohomological field theory is that the triple $(\mathcal{A}, \mathcal{G}, \mathbb{E})$ can be all infinite dimensional but certain path integral can still be reduced to an integral over finite dimensional space \mathcal{M} .

In Donaldson-Witten theory \mathcal{A} is the space of all connections (gauge fields) on a Hermitian vector bundle $E \rightarrow M$ over a complex 2-dimensional Kähler manifold M with Kähler form ω and \mathcal{G} is the group of all gauge transformations. This determines a localization equation from the momentum map μ ;

$$iF \wedge \omega - \frac{\zeta}{2}\omega^2 I_E = 0,\tag{2.2}$$

The solution space of this equation modulo \mathcal{G} is infinite dimensional. Thus we consider an infinite dimensional bundle $\mathbb{E} \rightarrow \mathcal{A}$ with \mathcal{G} -equivariant holomorphic section \mathfrak{S} . We introduce a complex structure of \mathcal{A} by declaring that the $A^{0,1}$ component of a connection 1-form $A = A^{1,0} + A^{0,1}$ represents holomorphic coordinates. Then there is an unique choice $\mathfrak{S} = F^{0,2}$ on a general Kähler manifold, leading to another localization equation,

$$F^{0,2} = 0.\tag{2.3}$$

An integrable connection $F^{0,2} = \bar{\partial}_A^2 = 0$ is called Einstein-Hermitian or Hermitian-Yang-Mills if it further satisfies (2.2). Thus the path integral is localized to the moduli space \mathcal{M}_{EH} of Hermitian-Yang-Mills connections, or equivalently, a result due to [26][27], the moduli space of semi-stable holomorphic bundles on M . In this case the anti-ghost bundle $\mathbb{V} \rightarrow \mathcal{M}$ is trivial, due to Donaldson, and the correlation functions of supersymmetric observables can be interpreted as certain intersection pairings of homology cycles in the moduli space \mathcal{M}_{EH} .

In the section we generalize Donaldson-Witten theory on Kähler surfaces. The basic idea is to extend our target space \mathcal{A} of the $N_c = (2, 0)$ model - corresponding to Donaldson-Witten theory, to the total space $T^*\mathcal{A}$ of the cotangent bundle of \mathcal{A} . Since \mathcal{A} is a flat affine Kähler manifold a cotangent vector is represented as an element of $\Omega^1(M, \text{End}(E))$. Thus we introduce additional bosonic fields φ given by an adjoint valued 1-form $\varphi \in \Omega^1(M, \text{End}(E))$. Then, by decomposing $\varphi = \varphi^{1,0} + \varphi^{0,1}$, we have to declare $\varphi^{1,0}$ to represent holomorphic coordinates on the fiber space of $T^*\mathcal{A}$, since we already fixed a complex structure of \mathcal{A} by declaring that the $A^{0,1}$ component of a connection 1-form represents holomorphic coordinates. A beautiful fact for any cotangent bundle of a Kähler manifold is that it always has canonical hyper-Kähler structure [28]. Thus it is natural to consider hyper-Kähler quotients of $T^*\mathcal{A}$ by \mathcal{G} ;

$$\begin{aligned} \partial_A^* \varphi^{1,0} &= 0, \\ i\Lambda(F + [\varphi^{1,0}, \varphi^{0,1}]) - \zeta I &= 0. \end{aligned} \tag{2.4}$$

However the resulting hyper-Kähler quotient space is infinite dimensional.² To obtain a finite dimensional space we extend the bundle $\mathbb{E} \rightarrow \mathcal{A}$ to $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$ and try to cut out the hyper-Kähler quotient space by the vanishing locus of suitable \mathcal{G} -equivariant holomorphic sections. A natural choice on a Kähler surface is

$$\begin{aligned} F^{0,2} &= 0, \\ \bar{\partial}_A \varphi^{1,0} &= 0, \\ \varphi^{1,0} \wedge \varphi^{1,0} &= 0, \end{aligned} \tag{2.5}$$

which defines Higgs bundles of Simpson [17][18]. The above equation can be viewed as a generalization of the integrability $\bar{\partial}_A^2 = 0$ of the connection $\bar{\partial}_A$ to the integrability of the extended connection $D'' = \bar{\partial}_A + \varphi^{1,0}$. Our model based on (2.5) is a generalization of Donaldson-Witten theory.

Another beautiful fact for any cotangent bundle of a Kähler manifold is that it always has the equivariant S^1 -action acting on the fiber. Such a S^1 -action on $T^*\mathcal{A}$ descends to the moduli spaces above. We will use the S^1 symmetry to define a family of models, which have many interesting limits.

2.1 Preliminaries

We consider a rank r Hermitian vector bundle $E \rightarrow M$ over a complex d -dimensional Kähler manifold M with Kähler form ω . Consider the space \mathcal{A}

²In one complex dimensions the hyper-Kähler quotient space is Hitchin's moduli space [29].

of all connections of E and the cotangent bundle $T^*\mathcal{A}$. First we determine the fields representing the cotangent space $T^*\mathcal{A}$. For the base space \mathcal{A} of $T^*\mathcal{A}$ we have connection 1-form $A = A^{1,0} + A^{0,1}$ with the usual gauge transformation law. We introduce a complex structure I on \mathcal{A} using the complex structure of M by declaring $A^{0,1}$ to represent holomorphic coordinates. Since \mathcal{A} is a flat affine Kähler manifold a cotangent vector is represented as an element of $\Omega^1(M, \text{End}(E))$. We introduce an adjoint valued bosonic 1-form $\varphi \in \Omega^1(M, \text{End}(E))$, which may be regarded as an element of the cotangent space of \mathcal{A} . According to the complex structure of M we have a decomposition $\varphi = \varphi^{1,0} + \varphi^{0,1}$. Then it is natural to fix the complex structure of the fiber space of $T^*\mathcal{A}$ by declaring $\varphi^{1,0}$ to be a holomorphic coordinate. Thus the (holomorphic) tangent space of $T^*\mathcal{A}$ is given by

$$\Omega^{0,1}(M, \text{End}(E)) \oplus \Omega^{1,0}(M, \text{End}(E)). \quad (2.6)$$

We denote the above complex structure also by I and call it the preferred complex structure, which has been induced from the complex structure of M . The total Kähler potential $\mathcal{K}(A, \varphi)$ of the total space $T^*\mathcal{A}$ is given by

$$\mathcal{K}(A, \varphi) = \mathcal{K}(A) - \frac{i}{2(d)!\pi^2} \int_M \text{Tr}(\varphi^{1,0} \wedge \varphi^{0,1}) \wedge \omega^{d-1} \quad (2.7)$$

where the Kähler potential $\mathcal{K}(A)$ of \mathcal{A} is

$$\mathcal{K}(A) = \frac{1}{4(d)!\pi^2} \int_M \kappa \text{Tr}(F \wedge F) \wedge \omega^{d-2}. \quad (2.8)$$

and the added term is a Kähler potential in the space \mathcal{B} . On the total space $T^*\mathcal{A}$ we have a obvious action of the infinite dimensional group \mathcal{G} of all gauge transformations, preserving the Kähler potential $\mathcal{K}(A, \varphi)$.

Now we introduce our $N_c = (2, 0)$ supercharges s_+ and \bar{s}_- with the familiar commutation relations

$$s_+^2 = 0, \quad \{s_+, \bar{s}_-\} = -i\phi_{++}^a \mathcal{L}_a, \quad \bar{s}_-^2 = 0. \quad (2.9)$$

The supercharges are identified with the differentials of \mathcal{G} -equivariant cohomology of our target space $T^*\mathcal{A}$. Thus $\phi_{++}^a \mathcal{L}_a$ is the infinitesimal gauge transformation generated by the adjoint scalar $\phi_{++} \in \text{Lie}(\mathcal{G}) = \Omega^0(M, \text{End}(E))$. From the complex structure of $T^*\mathcal{A}$ introduced above we have two sets of holomorphic multiplets $(A^{0,1}, \psi_+^{0,1})$ and $(\varphi^{1,0}, \lambda_+^{1,0})$ and their anti-holomorphic

partners. The supersymmetry transformation laws are given by

$$\begin{aligned} s_+ A^{0,1} &= i\psi_+^{0,1}, & s_+ \psi_+^{0,1} &= 0, \\ \bar{s}_+ A^{0,1} &= 0, & \bar{s}_+ \psi_+^{0,1} &= -\bar{\partial}_A \phi_{++}, \\ s_+ A^{1,0} &= 0, & s_+ \bar{\psi}_+^{1,0} &= -\partial_A \phi_{++}, \\ \bar{s}_+ A^{1,0} &= i\bar{\psi}_+^{1,0}, & \bar{s}_+ \bar{\psi}_+^{1,0} &= 0, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} s_+ \varphi^{1,0} &= i\lambda_+^{1,0}, & s_+ \lambda_+^{1,0} &= 0, \\ \bar{s}_+ \varphi^{1,0} &= 0, & \bar{s}_+ \lambda_+^{1,0} &= [\phi_{++}, \varphi^{1,0}], \\ s_+ \varphi^{0,1} &= 0, & s_+ \bar{\lambda}_+^{0,1} &= [\phi_{++}, \varphi^{0,1}], \\ \bar{s}_+ \varphi^{0,1} &= i\bar{\lambda}_+^{0,1}, & \bar{s}_+ \bar{\lambda}_+^{0,1} &= 0. \end{aligned} \tag{2.11}$$

From the transformation laws we have the following total \mathcal{G} -equivariant Kähler form on $T^*\mathcal{A}$,

$$\begin{aligned} \widehat{\varpi}_T^{\mathcal{G}} &= i s_+ \bar{s}_+ \mathcal{K}(A, \varphi) \\ &= \frac{i}{2(d)!\pi^2} \int_M \text{Tr} \left(\phi_{++} \left(F + [\varphi^{1,0}, \varphi^{0,1}] \right) \right) \wedge \omega^{d-1} \\ &\quad + \frac{1}{2(d)!\pi^2} \int_M \text{Tr} \left(\psi_+^{0,1} \wedge \bar{\psi}_+^{1,0} + \lambda_+^{1,0} \wedge \bar{\lambda}_+^{0,1} \right) \wedge \omega^{d-1}. \end{aligned} \tag{2.12}$$

The second term in the above is the Kähler form ϖ and the first term is the real \mathcal{G} -momentum map $\phi_{++}^a + \mu_a$, $\mu_{\mathbb{R}} : T^*\mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^* = \Omega^{2n}(M, \text{End}(E))$;

$$\begin{aligned} \mu_{\mathbb{R}} &= \frac{1}{2(d)!\pi^2} \left(F + [\varphi^{1,0}, \varphi^{0,1}] \right) \wedge \omega^{d-1} \\ &= \frac{1}{2d(d)!\pi^2} \Lambda \left(F + [\varphi^{1,0}, \varphi^{0,1}] \right) \omega^d, \end{aligned} \tag{2.13}$$

where Λ denote the adjoint of wedge multiplication with ω .

Following Hitchin [29] we have a natural hyper-Kähler structure I , J and K on $T^*\mathcal{A}$. Note that the additional complex structures J and K have no relation with the complex structure on the manifold M . Then we define the holomorphic symplectic form $\varpi_{\mathbb{C}}$ on $T^*\mathcal{A}$ by

$$\begin{aligned} \varpi_{\mathbb{C}}((\delta_1 A^{0,1}, \delta_1 \varphi^{1,0}), (\delta_2 A^{0,1}, \delta_2 \varphi^{1,0})) \\ = \frac{1}{2(d)!\pi^2} \int_M \text{Tr} \left(\delta_2 \varphi^{1,0} \wedge * \delta_1 A^{0,1} - \delta_1 \varphi^{1,0} \wedge * \delta_2 A^{0,1} \right). \end{aligned} \tag{2.14}$$

The corresponding complex momentum map $\mu_{\mathbb{C}}$ on $T^*\mathcal{A}$ is given by

$$\mu_{\mathbb{C}} = \frac{1}{2(d)!\pi^2} \bar{\partial}_A \varphi^{1,0} \wedge \omega^{d-1} = \frac{1}{2d \cdot (d)!\pi^2} (\Lambda \bar{\partial}_A \varphi^{1,0}) \wedge \omega^d. \quad (2.15)$$

Using the Kähler identities

$$\bar{\partial}_A^* = i[\partial_A, \Lambda], \quad \partial_A^* = -i[\bar{\partial}_A, \Lambda], \quad (2.16)$$

we see that the zeros of the complex momentum map is given by

$$\Lambda \bar{\partial}_A \varphi^{1,0} \rightarrow \partial_A^* \varphi^{1,0} = 0. \quad (2.17)$$

We again consider a rank r Hermitian vector bundle $E \rightarrow M$ over a complex 2-dimensional Kähler manifold M with Kähler form ω . Consider the space \mathcal{A} of all connections on E and the cotangent bundle $T^*\mathcal{A}$. We have the same holomorphic coordinates fields $A^{0,1}$ and $\varphi^{1,0} \in \Omega^{1,0}(M, \text{End}(E))$ of $T^*\mathcal{A}$, with the supersymmetry transformation laws in (2.10) and (2.11). We also have the usual $N_c = (2, 0)$ gauge multiplet.

2.2 Our Model

Now consider an infinite dimensional \mathcal{G} -equivariant holomorphic Hermitian vector bundle $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$ over $T^*\mathcal{A}$ with a suitable \mathcal{G} -equivariant holomorphic section $\tilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0})$, i.e., $\bar{s}_+ \tilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0}) = 0$. We only have the following possibility for this;

$$\tilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0}) = F^{0,2} \oplus \bar{\partial}_A \varphi^{1,0} \oplus (\varphi^{1,0} \wedge \varphi^{1,0}). \quad (2.18)$$

We choose this most general form as our holomorphic section. We have a natural paring of the holomorphic section with corresponding anti-ghost fields Υ_- given by $\int_M \text{Tr}(\Upsilon_- \wedge * \mathfrak{S})$. Thus the anti-ghost for the $F^{0,2}$ bit of section belongs to $\Omega^{2,0}(M, \text{End}(E))$, the anti-ghost for the mixed part belongs to $\Omega^{1,1}(M, \text{End}(E))$ and the anti-ghost for $(\varphi^{1,0} \wedge \varphi^{1,0})$ belongs to $\Omega^{0,2}(M, \text{End}(E))$. Associated with the holomorphic section $F^{0,2}$ over the base space \mathcal{A} of $T^*\mathcal{A}$ we have Fermi multiplet $(\chi_-^{2,0}, H^{2,0}) \in \Omega^{2,0}(M, \text{End}(E))$ and anti-Fermi multiplet $(\bar{\chi}_-^{0,2}, H^{0,2})$,

$$\begin{aligned} s_+ \chi_-^{2,0} &= -H^{2,0}, & s_+ H^{2,0} &= 0, \\ \bar{s}_+ \chi_-^{2,0} &= 0, & \bar{s}_+ H^{2,0} &= -i[\phi_{++}, \chi_-^{2,0}], \\ s_+ \bar{\chi}_-^{0,2} &= 0, & s_+ H^{0,2} &= -i[\phi_{++}, \bar{\chi}_-^{0,2}], \\ \bar{s}_+ \bar{\chi}_-^{0,2} &= -H^{0,2}, & \bar{s}_+ H^{0,2} &= 0. \end{aligned} \quad (2.19)$$

Associated with the mixed component of holomorphic section $\bar{\partial}_A \varphi^{1,0}$ over \mathcal{B} of $T^* \mathcal{A}$ we have Fermi multiplets $(\chi_-^{1,1}, H^{1,1}) \in \Omega^{1,1}(M, \text{End}(E))$ and their anti-Fermi partners $(\bar{\chi}_-^{1,1}, \bar{H}^{1,1})$,

$$\begin{aligned} s_+ \chi_-^{1,1} &= -H^{1,1}, & s_+ H^{1,1} &= 0, \\ \bar{s}_+ \chi_-^{1,1} &= 0, & \bar{s}_+ H^{1,1} &= -i[\phi_{++}, \chi_-^{1,1}], \\ s_+ \bar{\chi}_-^{1,1} &= 0, & s_+ \bar{H}^{1,1} &= -i[\phi_{++}, \bar{\chi}_-^{1,1}], \\ \bar{s}_+ \bar{\chi}_-^{1,1} &= -\bar{H}^{1,1}, & \bar{s}_+ \bar{H}^{1,1} &= 0. \end{aligned} \tag{2.20}$$

Associated with the holomorphic section $\varphi^{1,0} \wedge \varphi^{1,0}$ over the fiber space \mathcal{B} of $T^* \mathcal{A}$ we have Fermi multiplet $(\eta_-^{0,2}, K^{0,2}) \in \Omega^{2,0}(M, \text{End}(E))$ and their anti-Fermi partner $(\bar{\eta}_-^{2,0}, H^{2,0})$

$$\begin{aligned} s_+ \eta_-^{0,2} &= -K^{2,0}, & s_+ K^{2,0} &= 0, \\ \bar{s}_+ \eta_-^{0,2} &= 0, & \bar{s}_+ K^{2,0} &= -i[\phi_{++}, \eta_-^{0,2}], \\ s_+ \bar{\eta}_-^{2,0} &= 0, & \bar{s}_+ K^{0,2} &= -i[\phi_{++}, \bar{\eta}_-^{2,0}], \\ \bar{s}_+ \bar{\eta}_-^{2,0} &= -K^{0,2}, & s_+ K^{0,2} &= 0. \end{aligned} \tag{2.21}$$

Now we consider the following $N_c = (2, 0)$ supersymmetric action functional

$$\begin{aligned} S = & \frac{s_+ \bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\phi_{--} \left(F \wedge \omega + [\varphi^{1,0}, \varphi^{0,1}] \wedge \omega + \frac{i\zeta}{2} \omega^2 I_E \right) + \eta_- * \bar{\eta}_- \right) \\ & + \frac{s_+ \bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\chi_-^{2,0} \wedge * \bar{\chi}_-^{0,2} + \chi_-^{1,1} \wedge * \bar{\chi}_-^{1,1} + \eta_-^{0,2} \wedge * \bar{\eta}_-^{2,0} \right) \\ & + \frac{i s_+}{4\pi^2} \int_M \text{Tr} \left(\chi_-^{2,0} \wedge * F^{0,2} + \chi_-^{1,1} \wedge * \bar{\partial}_A \varphi^{1,0} + \eta_-^{0,2} \wedge * (\varphi^{1,0} \wedge \varphi^{1,0}) \right) \\ & + \frac{i \bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\bar{\chi}_-^{0,2} \wedge * F^{2,0} + \bar{\chi}_-^{1,1} \wedge * \partial_A \varphi^{0,1} + \bar{\eta}_-^{2,0} \wedge * (\varphi^{0,1} \wedge \varphi^{0,1}) \right), \end{aligned} \tag{2.22}$$

We set $\zeta = 0$ for simplicity by restricting to the case with $c_1(E) = 0$. By expanding the above and integrating out the auxiliary fields we see that the path integral is localized to the moduli space defined by the following equations

$$\begin{aligned} F^{0,2} &= 0, \\ \varphi^{1,0} \wedge \varphi^{1,0} &= 0, \\ \bar{\partial}_A \varphi^{1,0} &= 0, \\ i\Lambda(F + [\varphi^{1,0}, \varphi^{0,1}]) - \zeta I_E &= 0. \end{aligned} \tag{2.23}$$

The first three equations above are from $\tilde{\mathfrak{S}} = 0$ and the last equation is from the total momentum map $\mu_{\mathbb{R}}$ (2.13). The Higgs bundle $(\bar{\partial}_A, \varphi^{1,0})$ of Simpson [17][18] is defined by the first three equations in (2.23), i.e. the equations in (2.5). Those equations can be regarded as integrability $(\mathbf{D}'')^2 = 0$ of the extended half "connection" $\mathbf{D}'' = \bar{\partial}_A + \varphi^{1,0}$. There is notion of semi-stable Higgs bundle and a theorem analogous to Donaldson-Uhlenbeck-Yau such that every semi-stable Higgs bundle $(E, \varphi^{1,0})$ has an Einstein-Hermitian metric;

$$i\Lambda(F + [\varphi^{1,0}, \varphi^{0,1}]) - \zeta I_E = 0. \quad (2.24)$$

Furthermore the extended connection is flat $\mathbf{D}' \circ \mathbf{D}'' + \mathbf{D}'' \circ \mathbf{D}' = 0$ if and only if $c_1(E, \varphi^{1,0}) = c_2(E, \varphi^{1,0}) = 0$. Thus the path integral of our model is localized to the moduli space of semi-stable Higgs bundles. We also have other bosonic localization equations, as usual

$$\begin{aligned} d_A \phi_{++} &= 0, \\ [\phi_{++}, B] &= 0, \\ [\phi_{++}, \phi_{--}] &= 0. \end{aligned} \quad (2.25)$$

If the connections are irreducible we have $\phi_{\pm\pm} = 0$ and \mathcal{G} acts freely on the solution space of (2.23). The resulting moduli space is then isomorphic to the moduli space of stable Higgs bundles. We denote the moduli space of semi-stable Higgs bundle by \mathcal{N} . Note that the moduli space \mathcal{N} contains the moduli space \mathcal{M} of semi-stable bundles, equivalently the moduli space of EH or anti-self-dual connections on a Kähler surface M .

From now on we set $\zeta = 0$ for simplicity.

2.3 Comparison with Donaldson-Witten Theory

At this point it is useful to compare with Donaldson-Witten theory. The path integral of Donaldson-Witten theory is localized to the moduli space \mathcal{M} of anti-self-dual connections defined by

$$\begin{aligned} (\bar{\partial}_A)^2 &= 0, \\ \Lambda(\partial_A \circ \bar{\partial}_A + \bar{\partial}_A \circ \partial_A) &= 0. \end{aligned} \quad (2.26)$$

Define $\mathbf{D}'' = \bar{\partial}_A + \varphi^{1,0}$ and $\mathbf{D}' = \partial_A + \varphi^{0,1}$. Our localization equations (2.23) can be written as

$$\begin{aligned} (\mathbf{D}'')^2 &= 0, \\ \Lambda(\mathbf{D}' \circ \mathbf{D}'' + \mathbf{D}'' \circ \mathbf{D}') &= 0. \end{aligned} \quad (2.27)$$

Similarly we can combine the superpartners of $A^{0,1}$ and $\varphi^{1,0}$, and the anti-ghosts ($\chi_-^{2,0}, \chi_-^{1,1}, \eta_-^{0,2}$). To see this let us define extended fields

$$\begin{aligned} \mathbf{A}^{(0,1)} &:= A^{0,1} + \varphi^{1,0}, & \mathbf{A}^{(1,0)} &:= A^{1,0} + \varphi^{0,1}, \\ \Psi_+^{(0,1)} &:= \psi_+^{0,1} + \lambda_+^{1,0}, & \overline{\Psi}_+^{(1,0)} &:= \overline{\psi}_+^{1,0} + \overline{\lambda}_+^{0,1}, \\ \Upsilon_-^{(2,0)} &:= \chi_-^{2,0} + \chi_-^{1,1} + \eta_-^{0,2}, & \overline{\Upsilon}_-^{(0,2)} &:= \overline{\chi}_-^{0,2} + \overline{\chi}_-^{1,1} + \overline{\eta}_-^{2,0}, \\ \mathbf{H}^{(2,0)} &:= H^{2,0} + H^{1,1} + K^{0,2}, & \mathbf{H}^{(0,2)} &:= H^{0,2} + \overline{H}^{1,1} + K^{2,0}, \end{aligned} \quad (2.28)$$

where the superscript of the extended fields represent a graded form degree on M . That is we exchange holomorphic and antiholomorphic differential form degree on M of fields associated with $\varphi^{1,0}$ and $\varphi^{0,1}$. For example the extended anti-ghost $\Upsilon_-^{(2,0)}$ is associated with the total holomorphic section $\tilde{\mathfrak{S}} := \mathbf{F}^{(0,2)} := F^{0,2} + \overline{\partial}_A \varphi^{1,0} + \varphi^{1,0} \wedge \varphi^{1,0}$ of $\tilde{\mathbb{E}} \rightarrow T^* \mathcal{A}$ by the pairing $\int_M \text{Tr}(\Upsilon_-^{(2,0)} \wedge * \mathbf{F}^{(0,2)})$. Note that the combinations (2.28) preserve the ghost numbers

$$\begin{aligned} \Psi_+^{(0,1)} &: (+1, 0), & \overline{\Psi}_+^{(1,0)} &: (0, +1), \\ \Upsilon_-^{(2,0)} &: (-1, 0), & \overline{\Upsilon}_-^{(0,2)} &: (0, -1). \end{aligned} \quad (2.29)$$

The supersymmetry transformation laws for the coordinate fields of $T^* \mathcal{A}$ are, combining (2.10) and (2.11),

$$\begin{aligned} s_+ \mathbf{A}^{(0,1)} &= i \Psi_+^{(0,1)}, & s_+ \Psi_+^{(0,1)} &= 0, \\ \overline{s}_+ \mathbf{A}^{(0,1)} &= 0, & \overline{s}_+ \Psi_+^{(0,1)} &= -\mathbf{D}'' \phi_{++}, \\ s_+ \mathbf{A}^{(1,0)} &= 0, & s_+ \overline{\Psi}_+^{(1,0)} &= -\mathbf{D}' \phi_{++}, \\ \overline{s}_+ \mathbf{A}^{(1,0)} &= i \overline{\Psi}_+^{(1,0)}, & \overline{s}_+ \overline{\Psi}_+^{(1,0)} &= 0. \end{aligned} \quad (2.30)$$

The supersymmetry transformation laws for the Fermi multiplet $(\Upsilon_-^{(2,0)}, \mathbf{H}^{(2,0)})$ are, by combining (2.19), (2.20), and (2.21) together,

$$\begin{aligned} s_+ \Upsilon_-^{(2,0)} &= -\mathbf{H}^{(2,0)}, & s_+ \mathbf{H}^{(2,0)} &= 0, \\ \overline{s}_+ \Upsilon_-^{(2,0)} &= 0, & \overline{s}_+ \mathbf{H}^{(2,0)} &= -i[\phi_{++}, \Upsilon_-^{(2,0)}], \\ s_+ \overline{\Upsilon}_-^{(0,2)} &= 0, & s_+ \mathbf{H}^{(0,2)} &= -i[\phi_{++}, \overline{\Upsilon}_-^{(0,2)}], \\ \overline{s}_+ \overline{\Upsilon}_-^{(0,2)} &= -\mathbf{H}^{(0,2)}, & \overline{s}_+ \mathbf{H}^{(0,2)} &= 0. \end{aligned} \quad (2.31)$$

We have the usual $N_c = (2, 0)$ gauge multiplet associated with the unitary gauge transformation. For convenience we rewrite down supersymmetry the

transformation laws

$$\begin{aligned}
s_+ \eta_- &= 0, \\
s_+ \phi_{--} &= i\eta_-, \quad \bar{s}_+ \eta_- = +iH_0 + \frac{1}{2}[\phi_{++}, \phi_{--}], \quad s_+ \phi_{++} = 0, \\
\bar{s}_+ \phi_{--} &= i\bar{\eta}_-, \quad s_+ \bar{\eta}_- = -iH_0 + \frac{1}{2}[\phi_{++}, \phi_{--}], \quad \bar{s}_+ \phi_{++} = 0. \\
\bar{s}_+ \bar{\eta}_- &= 0,
\end{aligned} \tag{2.32}$$

Now the action functional S in (2.22) can be written as

$$\begin{aligned}
S = & \frac{s_+ \bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\phi_{--} \mathbf{F} \right) \wedge \omega \\
& + \frac{s_+ \bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\eta_- \wedge * \bar{\eta}_- + \Upsilon_-^{(2,0)} \wedge * \bar{\Upsilon}_-^{(0,2)} \right) \\
& + \frac{is_+}{4\pi^2} \int_M \text{Tr} \left(\Upsilon_-^{(2,0)} \wedge * \mathbf{F}^{(0,2)} \right) + \frac{i\bar{s}_+}{4\pi^2} \int_M \text{Tr} \left(\bar{\Upsilon}_-^{(0,2)} \wedge * \mathbf{F}^{(2,0)} \right),
\end{aligned} \tag{2.33}$$

where

$$\mu_{\mathbb{R}} = \frac{1}{4\pi^2} \mathbf{F} \wedge \omega. \tag{2.34}$$

The above action functional has exactly same form as Donaldson-Witten theory on Kähler 2-folds. We remark that the Kähler identities (2.16) are important technical tools in analyzing Donaldson-Witten theory on Kähler manifolds. Simpson showed that one also has the Kähler identities for Higgs bundles,

$$(\mathbf{D}')^* = i[\Lambda, \mathbf{D}''], \quad (\mathbf{D}'')^* = -i[\Lambda, \mathbf{D}'], \tag{2.35}$$

We will work with the above shorthand notations.

2.4 The Path Integral and New Invariants of Four-Manifolds

The explicit form of the total action functional S' after integrating out all the auxiliary fields from S is given by

$$\begin{aligned}
S' = & \frac{1}{4\pi^2} \int_M \text{Tr} \left(-\frac{1}{2} \mathbf{F}^+ \wedge * \mathbf{F}^+ - \mathbf{D} \phi_{++} * \mathbf{D} \phi_{--} + \frac{1}{4} [\phi_{++}, \phi_{--}] * [\phi_{++}, \phi_{--}] \right. \\
& + \phi_{--} * \Lambda[\Psi_+^{(0,1)}, \bar{\Psi}_+^{(1,0)}] + i\Upsilon_-^{2,0} \wedge * [\phi_{++}, \bar{\Upsilon}_-^{(0,2)}] + i[\phi_{++}, \eta_-] * \bar{\eta}_- \\
& - i\mathbf{D}' \bar{\eta}_- \wedge * \Psi_+^{(0,1)} - i\mathbf{D}'' \eta_- \wedge * \bar{\Psi}_+^{(1,0)} - \Upsilon_-^{2,0} \wedge * \mathbf{D}'' \Psi_+^{(0,1)} \\
& \left. - \bar{\Upsilon}_-^{(0,2)} \wedge * \mathbf{D}' \bar{\Psi}_+^{(1,0)} \right),
\end{aligned} \tag{2.36}$$

where $\mathbf{D} = \mathbf{D}' + \mathbf{D}''$ and we used the extended Kähler identities (2.35). We also used notation \mathbf{F}^+ , which is given by

$$\mathbf{F}^+ = \mathbf{F}^{(2,0)} + \frac{1}{2}(\Lambda \mathbf{F})\omega + \mathbf{F}^{(0,2)}, \quad (2.37)$$

so that $\mathbf{F}^+|_{\varphi^{1,0}=\varphi^{0,1}=0} = F^+$, where F^+ denotes the self-dual part of the standard curvature two-form. Note that \mathbf{F}^+ also contains anti-self-dual two-form part as well.

Now we examine the equations for fermionic zero-modes. The equation of motions for fermions are, modulo infinitesimal gauge transformations,

$$\begin{aligned} i\mathbf{D}''\bar{\eta}_- + (\mathbf{D}'')^*\bar{\Upsilon}_-^{(0,2)} &= 0, \\ (\mathbf{D}'')^*\Psi_+^{(0,1)} &= 0, \\ \mathbf{D}''\Psi_+^{(0,1)} &= 0. \end{aligned} \quad (2.38)$$

Using one of the bosonic localization equation $(\mathbf{D}'')^2 = 0$, we find that the fermionic zero-modes are governed by the following equations

$$\begin{aligned} \mathbf{D}''\bar{\eta}_- &= 0, & (\mathbf{D}'')^*\Psi_+^{(0,1)} &= 0, & (\mathbf{D}'')^*\bar{\Upsilon}_-^{(0,2)} &= 0. \end{aligned} \quad (2.39)$$

Thus the fermionic zero-modes are elements of cohomology group of the following extended Dolbeault complex

$$0 \rightarrow \mathbf{S}^{(0,0)} \xrightarrow{\mathbf{D}''} \mathbf{S}^{(0,1)} \xrightarrow{\mathbf{D}''} \mathbf{S}^{(0,2)} \rightarrow 0, \quad (2.40)$$

where

$$\mathbf{S}^{(0,p)} = \bigoplus_{r+s=p} \Omega^{0,r}(M, \wedge^s(T_M^{*1,0}) \otimes \text{End}(E)). \quad (2.41)$$

The net ghost number violation in the path integral measure due to fermionic zero-modes is $(\tilde{\Delta}, \tilde{\Delta})$ where $\tilde{\Delta}$ is the negative of the index of the above complex. Almost all of the standard procedure in Donaldson-Witten theory can be repeated here. For example observables are \mathcal{G} -equivariant closed differential forms, after the parity change, on the space $T^*\mathcal{A}$. As for a canonical observable we have the \mathcal{G} -equivariant Kähler form, after the parity change, on $T^*\mathcal{A}$;

$$\widehat{\varpi}_T^{\mathcal{G}} = \frac{i}{4\pi^2} \int_M \text{Tr} \left(\phi_{++} \mathbf{F} \right) \wedge \omega + \frac{1}{4\pi^2} \int_M \text{Tr} \left(\Psi_+^{(0,1)} \wedge \bar{\Psi}_+^{(1,0)} \right) \wedge \omega, \quad (2.42)$$

The correlation functions of supersymmetric observables are the path integral representations of a generalized Donaldson-Witten invariant.

We note that the fundamental group of four-manifold does not seem to play any essential roles in the original Donaldson-Witten theory. On the other hand the most crucial application of Simpson's Higgs bundle is on the non-Abelian Hodge theory associated with the representation variety $\pi_1(M) \rightarrow GL(r, \mathbb{C})$ of the fundamental group. For this purpose let us consider the case the $c_1(E) = c_2(E) = 0$.³ It is known that there is a one-to-one correspondence between irreducible representations of $\pi_1(M)$ and stable Higgs bundles with vanishing Chern classes, see [18]. In this situation Donaldson-Witten invariants concern only the unitary irreducible representation variety. An important property of the moduli space of stable Higgs bundles is the existence of a \mathbb{C}^* action $(E, \varphi^{1,0}) \rightarrow (E, t\varphi^{1,0})$. Simpson showed that the fixed points of \mathbb{C}^* action correspond to complex variations of Hodge structures. It also implies that any other representation of $\pi_1(M)$ can be deformed to a complex variation of Hodge structures. Among the fixed points the trivial complex variation of Hodge structures corresponds to unitary irreducible representations. A useful viewpoint of the \mathbb{C}^* action is to regard it as Hodge decomposition of non-Abelian cohomology. Then a unitary representation is some kind of zero-form. The above results also imply that the path integral of our model for $c_1(E) = c_2(E) = 0$ can be written as the sum of contributions from every complex variation of Hodge structures. Thus it is natural to hope that our new invariants may have information beyond the Donaldson-Witten and Seiberg-Witten invariants for non-simply connected Kähler 2-folds. Of course we do not need to restrict our attention to the flat case.

The moduli space of stable Higgs bundles have many beautiful properties and applications. One of the properties is that the rank r stable Higgs sheaves on M can be identified with stable sheaves on the cotangent bundle T^*M which are supported on Lagrangian subvarieties of T^*M which are finite degree r branched coverings of M [19][31]. The above property may be relevant to generalized mirror symmetry on Calabi-Yau 4-folds [32][33]. If we consider the complex 2-torus, T^4 , its cotangent bundle may be regarded as local model for T^4 -fibered Calabi-Yau 4-folds. Then the moduli space of stable rank r Higgs sheaves may be viewed as parameterizing r $D4$ -branes wrapped on Lagrangian cycles of Calabi-Yau 4-folds. Of course the above

³It is not obvious if the moduli space of stable Higgs bundles has a hyper-Kähler structure. For the flat case the existence of hyper-Kähler structure has proved by Fujiki [30].

picture is too naive but somewhat suggestive. Here we will not be able to penetrate many of the applications and properties of Higgs bundles. We will use its S^1 symmetry to have an anatomy of our invariants.

2.5 Flows to Donaldson-Witten theory

In the laymen's terms Donaldson-Witten invariant is simply the symplectic volume of the moduli space \mathcal{M} of stable bundles on M . Similarly, the invariants defined by the correlation function $\langle \exp(\hat{\omega}_T^G) \rangle$ is the symplectic volume of the moduli space \mathcal{N} of stable Higgs bundles. One of most important properties of the moduli space \mathcal{N} is that it has a symmetry under a S^1 -action, which can be extended to a \mathbb{C}^* -action. The beautiful fact is that the \mathbb{C}^* action is a very special one, related with a certain variation of Hodge structures⁴

First we note that our localization equations in (2.23) are more than the equations (2.27). We may replace \mathbf{D}'' by a family of extended derivatives by introducing a spectral parameter t ,

$$\begin{aligned} \mathbf{D}'' &= \bar{\partial}_A + t\varphi^{1,0}, \\ \mathbf{D}' &= \partial_A + \bar{t}\varphi^{0,1}. \end{aligned} \tag{2.43}$$

Then our localization equations in (2.23) imply that

$$\begin{aligned} (\mathbf{D}'')^2 &= 0, \\ \Lambda(\mathbf{D}' \circ \mathbf{D}'' + \mathbf{D}'' \circ \mathbf{D}') &= 0, \end{aligned} \tag{2.44}$$

for any t with $t\bar{t} = 1$. Similarly we replace the extended fields defined in (2.28) as follows

$$\begin{aligned} \mathbf{A}^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & \mathbf{A}^{(1,0)} &:= A^{1,0} + \bar{t}\varphi^{0,1}, \\ \Psi_+^{(0,1)} &:= \psi_+^{0,1} + t\lambda_+^{1,0}, & \bar{\Psi}_+^{(1,0)} &:= \bar{\psi}_+^{1,0} + \bar{t}\lambda_+^{0,1}, \\ \Upsilon_-^{(2,0)} &:= \chi_+^{2,0} + \bar{t}\chi_-^{1,1} + \bar{t}^2\eta_-^{0,2}, & \bar{\Upsilon}_-^{(0,2)} &:= \bar{\chi}_-^{0,2} + t\bar{\chi}_-^{1,1} + t^2\bar{\eta}_-^{2,0}, \\ \mathbf{H}^{(2,0)} &:= H^{2,0} + \bar{t}H^{1,1} + \bar{t}^2K^{0,2}, & \mathbf{H}^{(0,2)} &:= H^{0,2} + t\bar{H}^{1,1} + t^2K^{2,0}. \end{aligned} \tag{2.45}$$

⁴This notion will be relevant to the case when the Higgs bundle is flat. Then $\mathbf{D} = \mathbf{D}' + \mathbf{D}''$ can be identified with the Gauss-Manin connections of the associated local system. Then our localization equations are familiar tt^* -equations in special geometry [34]. In fact for any complex, not necessarily integral, variation of Hodge structures there is a corresponding flat Higgs bundle.

Then our action functional S in (2.22) or (2.33) is invariant for any t with $t\bar{t} = 1$.

We will show shortly that the S^1 action can be extended to a \mathbb{C}^* action by "gauging" the $U(1) = S^1$ symmetry and scaling the unit $U(1)$ charge. Such a procedure is equivalent to giving physical bare mass m to the $U(1)$ charged fields. Thus one can consider an imaginary \mathbb{CP}^1 where the \mathbb{C}^* action covers the natural \mathbb{C}^* action on \mathbb{CP}^1 with limit points ($t = 0, t = \infty$). Now we can identify the two limit points in \mathbb{CP}^1 with $(m = \infty, m = 0)$. Thus we can interpret the absolute flow generated by the \mathbb{C}^* action as a renormalization group flow from the past or unbroken phase $m = 0$ to the future (present) or broken phase $m \rightarrow \infty$. This is not just a mere fantasy since we indeed have a twistor space constructed from the function space of fields namely the total space $T^*\mathcal{A}$ of the cotangent bundle over the space of all gauge fields. Our field space has a hyper-Kähler structure preserved by the \mathcal{G} as well as by the S^1 symmetry acting on the fiber of $T^*\mathcal{A}$. Such a S^1 action can be extended to a \mathbb{C}^* action and then cover the \mathbb{C}^* action of \mathbb{CP}^1 in the twistor space $T^*\mathcal{A} \times \mathbb{CP}^1$. Furthermore the Hamiltonian of the S^1 -action on the field space is precisely the physical bare mass of the bosonic fields, whose field space are the fiber of $T^*\mathcal{A}$ on space-time M . Now by taking the $m \rightarrow \infty$ limit the dominant contributions to path integral come from the critical points of the Hamiltonian, equivalently from the fixed points of S^1 -action. Similarly in the $t \rightarrow 0$ limit any point in the field space flows to a certain fixed point of the S^1 -action. In the trivial fixed point $\varphi^{1,0} = 0$ we recover original Donaldson-Witten theory. As a global supersymmetric field theory on M certain path integral of our model will be localized to a finite dimensional subspace \mathcal{N} of the hyper-Kähler quotient of $T^*\mathcal{A}$ by \mathcal{G} . The above argument is valid regardless whether \mathcal{N} preserves the hyper-Kähler structure or not.

We may ask an interesting physical question. Donaldson-Witten theory is the twisted $N = 2$ supersymmetric Yang-Mills theory. On a manifold with trivial canonical line bundle twisting does nothing and we have space-time supersymmetric Yang-Mills theory. Then where shall we place our model? Our proposal is that it may describe a certain unbroken phase of bigger symmetry which is connected to the physical super-Yang-Mills theory by renormalization group flows, and the physical theory lives in one of the fixed points.

Now we perturbed our model by "gauging" the $U(1)$ symmetry. For this we modify the supersymmetry transformation laws according to the

following anti-commutations relations

$$s_+^2 = 0, \quad \{s_+, \bar{s}_+\} = -i\phi_{++}^a \mathcal{L}_a - im\mathcal{L}_{S^1}, \quad \bar{s}_+^2 = 0. \quad (2.46)$$

We define a new action functional $S(m)$ by the same formula as S in (2.33) but with the modified transformation laws. Then we define a family of $N_c = (2,0)$ models parameterized by m and \bar{m} with the following action functional

$$S(m, \bar{m}) = S(m) + i\bar{m}s_+\bar{s}_+\mathcal{K}(\mathbf{D}), \quad (2.47)$$

where $\mathcal{K}(\mathbf{D})$ is the Kähler potential of $T^*\mathcal{A}$ given by (2.7). Then the action functional contains bare mass terms for all the charged fields under the $U(1)$, except for auxiliary fields. The relevant terms in the action functional looks like

$$\begin{aligned} S(m, \bar{m}) = & S \\ & - \frac{\bar{m}}{4\pi^2} \int_M \text{Tr} \left(i\phi_{++}(F + [\varphi^{1,0}, \varphi^{0,1}]) + \psi_+^{0,1} \wedge \bar{\psi}_+^{1,0} + \lambda_+^{1,0} \wedge \bar{\lambda}_+^{0,1} \right) \wedge \omega \\ & + \frac{im\bar{m}}{4\pi^2} \int_M \text{Tr} \left(\varphi^{1,0} \wedge \varphi^{0,1} \right) \wedge \omega + \dots \end{aligned} \quad (2.48)$$

In the above the $m\bar{m}$ dependent term is the Hamiltonian of the S^1 -action on $T^*\mathcal{A}$. The term in the second line is the equivariant Kähler form $\widehat{\omega}_T^G$ of $T^*\mathcal{A}$. Thus $\widehat{\omega}^G := \widehat{\omega}_T^G|_{\mathcal{A}}$ is an observable of Donaldson-Witten theory which will descend to the Kähler form of moduli space \mathcal{M} of anti-self-dual connections.

Now by taking the $m \rightarrow \infty$ limit we see that the dominant contributions to the path integral come from the critical points of the Hamiltonian of the S^1 -action. Such critical points are identical to the fixed points of the S^1 -action. As usual we always have trivial fixed points given by $\varphi^{1,0} = 0$ and the fixed point locus is the moduli space \mathcal{M} of anti-self-dual connections. Thus the contribution from the trivial fixed points to the partition function of the model with the action functional $S(m, \bar{m})$ is given by a generating functional $\langle \exp(\widehat{\omega}^G) \rangle_{DW}$ of Donaldson-Witten theory weighted by one loop contributions from the degrees of freedom normal to \mathcal{M} in \mathcal{N} . We also note that the value of the Hamiltonian of the S^1 -action at the trivial fixed point is zero. There are other non-trivial fixed points $\varphi^{1,0} \neq 0$ if the S^1 -action can be undone by the gauge transformations,

$$g\varphi^{1,0}g^{-1} = t\varphi^{1,0}, \quad (2.49)$$

where $g \in \mathcal{G}$ and $t \in U(1)$.

3 Generalized Vafa-Witten Theory

A class of equivariant $N_c = (2, 0)$ sigma model in zero dimensions can be extended to a $N_c = (2, 2)$ model [22].⁵ The essential point of such a construction is introducing additional bosonic fields corresponding to the local frame fields on the image of the section $\mathfrak{S} : \mathcal{A} \rightarrow \mathbb{E}$ so that one has supersymmetric sigma model in zero-dimension with the target space being the total space of the bundle $\mathbb{E} \rightarrow \mathcal{A}$. Then the \mathcal{G} -equivariant holomorphic section \mathfrak{S} of $\mathbb{E} \rightarrow \mathcal{A}$ should be extended to gradient vector of a \mathcal{G} -invariant holomorphic function of the total space of the bundle $\mathbb{E} \rightarrow \mathcal{A}$.

Such $N_c = (2, 2)$ extension of Donaldson-Witten theory corresponds to Vafa-Witten theory on a Kähler surface [39]. Then one can define a family of $N_c = (2, 0)$ models by certain massive perturbation using a natural S^1 symmetry of the $N_c = (2, 2)$ model, see also [20][40][41]. By taking the bare mass to infinity one see that there are two different semi-classical limits corresponding to Donaldson-Witten and Seiberg-Witten theories. Then the S -duality of $N = 4$ SYM theory [20][5] implies the equivalence between Donaldson-Witten and Seiberg-Witten invariants [39][42].

In this section we apply the above construction to embed the $N_c = (2, 0)$ model in the previous section a $N_c = (2, 2)$ model. The resulting model generalizes Vafa-Witten theory and compute Euler characteristic of the moduli space of stable Higgs bundles together with extra contributions. Then we define \mathbb{C}^* family of $N_c = (2, 2)$ models which has various limits, which includes the original Vafa-Witten theory. We also perturb the model to $N_c = (2, 0)$ model and show that the partition function of the theory is sum of contributions of the generalized Donaldson-Witten theory in the previous section and a generalized version of Seiberg-Witten theory.

3.1 The $N_c = (2, 2)$ Model

We recall the basic setting for the previous $N_c = (2, 0)$ model. We considered the total space $T^*\mathcal{A}$ of the cotangent bundle of the space of all connections of a rank r Hermitian vector bundle $E \rightarrow M$ over a Kähler surface M . As for

⁵A $N_c = (2, 2)$ (or $N_c = (2, 0)$) model can be viewed as the dimensional reduction of $N_{ws} = (2, 2)$ (or $N_{ws} = (2, 0)$) world-sheet supersymmetric gauged linear sigma model in two-dimensions [35]. In the present case both the target space and the symmetry group are infinite dimensional. Related examples can be found in [36]. In our terminology a cohomological field theory [37] is an equivariant $N_c = (1, 0)$ sigma model in zero dimensions, while a balanced cohomological field theory [38] is an equivariant $N_c = (1, 1)$ sigma model in zero dimensions. With a Kähler structure on the target space the number of global fermionic symmetry is doubled.

the holomorphic coordinate fields on $T^*\mathcal{A}$ we have the extended connection $\mathbf{A}^{0,1}$ with superpartner $\Psi_+^{0,1}$. We also considered an infinite dimensional \mathcal{G} -equivariant holomorphic vector bundle $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$ with holomorphic section $\mathfrak{S}(\mathbf{D}'') = (\mathbf{D}'')^2 := \mathbf{F}^{(0,2)}$ and associated anti-ghost multiplet $(\Upsilon_-^{(2,0)}, \mathbf{H}^{(2,0)})$.

The basic idea behind the extension to a $N_c = (2, 2)$ model is that one can regard the total space of the holomorphic bundle $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$ as the target space of a $N_c = (2, 2)$ model. Then we have to supply local holomorphic coordinate fields for fiber space of $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$. Thus we introduce adjoint-valued bosonic spectral fields $\mathbf{B}^{(2,0)}$ and its superpartner $\Upsilon_+^{(2,0)}$. Now the former holomorphic section $\tilde{\mathfrak{S}} = \mathbf{F}^{(0,2)}(\mathbf{D}'')$ of the bundle $\tilde{\mathbb{E}} \rightarrow T^*\mathcal{A}$ corresponds to a holomorphic vector field on the target space $\tilde{\mathbb{E}}$ but being supported only on $T^*\mathcal{A}$. Thus the \mathcal{G} -equivariant holomorphic vector $\mathfrak{S}(\mathbf{D}'')$ should be extended over the whole space $\tilde{\mathbb{E}}$. Furthermore $N_c = (2, 2)$ supersymmetry demands that a such holomorphic vector should be the gradient vector of a non-degenerated \mathcal{G} -invariant holomorphic function \mathcal{W} , i.e., $\bar{s}_+ \mathcal{W} = 0$, on the target space $\tilde{\mathbb{E}}$.

Now demanding $N_c = (2, 2)$ supersymmetry will take care of everything. From the $N_c = (2, 0)$ holomorphic multiplets $(\mathbf{A}^{(0,1)}, \Psi_+^{(0,1)})$ we build up the following chiral multiplets, i.e., $\bar{s}_\pm \mathbf{A}^{(0,1)} = 0$

$$\begin{array}{ccccc} \Psi_-^{(0,1)} & \xleftarrow{s_-} & \mathbf{A}^{(0,1)} & \xrightarrow{s_+} & \Psi_+^{(0,1)} \\ & \searrow s_+ & & \swarrow s_- & \\ & & \mathbf{H}^{(0,1)} & & \end{array} \quad (3.1)$$

From the $N_c = (2, 0)$ Fermi multiplets $(\Upsilon_-^{(2,0)}, \mathbf{H}^{(2,0)})$ we build up another set of chiral multiplets, i.e., $\bar{s}_\pm \mathbf{B}^{(2,0)} = 0$

$$\begin{array}{ccccc} \Upsilon_-^{(2,0)} & \xleftarrow{s_-} & \mathbf{B}^{(2,0)} & \xrightarrow{s_+} & \Upsilon_+^{(2,0)} \\ & \searrow s_+ & & \swarrow s_- & \\ & & \mathbf{H}^{(2,0)} & & \end{array} \quad (3.2)$$

Form the $N_c = (2, 0)$ gauge multiplet $(\phi_{--}, \eta_-, \bar{\eta}_-, H_0, \phi_{++})$ we build up a $N_c = (2, 2)$ gauge multiplet

$$\begin{array}{ccccc} \bar{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\ \downarrow \bar{s}_- & & \downarrow \bar{s}_- & & \downarrow \bar{s}_- \\ \bar{\eta}_- & \xrightarrow{s_+} & H_0 & \xleftarrow{s_-} & \bar{\eta}_+ , \\ \uparrow \bar{s}_+ & & \uparrow \bar{s}_+ & & \uparrow \bar{s}_+ \\ \phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma \end{array} \quad (3.3)$$

which are adjoint valued scalars on M .

To keep track of all the fields we write down the explicit spectral form of the extended fields. We have

$$\begin{aligned} \mathbf{A}^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & \mathbf{A}^{(1,0)} &:= A^{1,0} + \bar{t}\varphi^{0,1}, \\ \Psi_{\pm}^{(0,1)} &:= \psi_{\pm}^{0,1} + t\lambda_{\pm}^{1,0}, & \bar{\Psi}_{\pm}^{(1,0)} &:= \bar{\psi}_{\pm}^{1,0} + \bar{t}\bar{\lambda}_{\pm}^{0,1}, \\ \mathbf{H}^{(1,0)} &:= H^{1,0} + tL^{1,0}, & \mathbf{H}^{(0,1)} &:= H^{0,1} + \bar{t}L^{0,1}, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \mathbf{B}^{(2,0)} &:= B^{2,0} + \bar{t}B^{1,1} + \bar{t}^2C^{0,2}, & \mathbf{B}^{(0,2)} &:= B^{0,2} + t\bar{B}^{1,1} + t^2C^{2,0}, \\ \Upsilon_{\pm}^{(2,0)} &:= \chi_{\pm}^{2,0} + \bar{t}\chi_{\pm}^{1,1} + \bar{t}^2\eta_{\pm}^{0,2}, & \bar{\Upsilon}_{\pm}^{(0,2)} &:= \bar{\chi}_{\pm}^{0,2} + t\bar{\chi}_{\pm}^{1,1} + t^2\bar{\eta}_{\pm}^{2,0}, \\ \mathbf{H}^{(2,0)} &:= H^{2,0} + \bar{t}H^{1,1} + \bar{t}^2K^{0,2}, & \mathbf{H}^{(0,2)} &:= H^{0,2} + t\bar{H}^{1,1} + t^2K^{2,0}. \end{aligned} \quad (3.5)$$

Now we have the standard $N_c = (2, 2)$ invariant functional

$$\begin{aligned} S = & s_+ \bar{s}_+ s_- \bar{s}_- \left(\mathcal{K}(\mathbf{D}) + \mathcal{K}(\mathbf{B}^{(2,0)}, \mathbf{B}^{(0,2)}) - \int_M \text{Tr}(\sigma * \bar{\sigma}) \right) \\ & + s_+ s_- \mathcal{W}(\mathbf{A}^{(0,1)}, \mathbf{B}^{(2,0)}) + \bar{s}_+ \bar{s}_- \bar{\mathcal{W}}(\mathbf{A}^{(1,0)}, \mathbf{B}^{(0,2)}), \end{aligned} \quad (3.6)$$

where

$$\mathcal{K}(\mathbf{B}^{(2,0)}, \mathbf{B}^{(0,2)}) = -\frac{1}{4\pi^2} \int_M \text{Tr}(\mathbf{B}^{(2,0)} \wedge * \mathbf{B}^{(0,2)}). \quad (3.7)$$

The holomorphic potential \mathcal{W} , i.e., $\bar{s}_{\pm} \mathcal{W} = 0$, is given as follows

$$\mathcal{W}(\mathbf{A}^{(0,1)}, \mathbf{B}^{(2,0)}) = \frac{1}{4\pi^2} \int_M \text{Tr}(\mathbf{B}^{(2,0)} \wedge * \mathbf{F}^{(0,2)}). \quad (3.8)$$

We note that the above action functional remains invariant for any t in (3.4) and (3.5) with $t\bar{t} = 1$. We will use this S^1 symmetry to define a \mathbb{C}^* family of the $N_c = (2, 2)$ model.

Now, from the discussions in Sect. 3.4, we see that the path integral is localized to the zeros of the momentum map $\mu_{\mathbb{R}}$ and the critical points of the holomorphic potential \mathcal{W} , modulo the gauge symmetry,

$$\begin{aligned} \mathbf{F}^{(0,2)} &= 0, \\ \mathbf{D}'' * \mathbf{B}^{(2,0)} &= 0, \\ i\mathbf{F} \wedge \omega + [\mathbf{B}^{(2,0)}, * \mathbf{B}^{(0,2)}] &= 0. \end{aligned} \quad (3.9)$$

We also have other default localization equations

$$\begin{aligned} [\bar{\sigma}, \mathbf{B}^{(0,2)}] &= 0, & [\phi_{\pm\pm}, \mathbf{B}^{(0,2)}] &= 0, \\ [\sigma, \mathbf{B}^{(0,2)}] &= 0, & [\phi_{\pm\pm}, \bar{\sigma}] &= 0, \\ [\sigma, \bar{\sigma}] &= 0, & [\phi_{++}, \phi_{--}] &= 0, \\ \mathbf{D}\bar{\sigma} &= 0, & \mathbf{D}\phi_{\pm\pm} &= 0. \end{aligned} \tag{3.10}$$

When there are no reducible orbits in (3.9) we have $\sigma = \phi_{\pm\pm} = 0$, and the path integral is localized to the moduli space defined by (3.9). The equation (3.9) is a generalization of Vafa-Witten equation. We note that the equations in (3.9), as well as in (3.10), remain the same for any t in (3.4) and (3.5) with $t\bar{t} = 1$, which is a symmetry of the action functional.

The equations in (3.9) have another S^1 symmetry given by

$$(\mathbf{D}'', \mathbf{B}^{(2,0)}) \rightarrow (\mathbf{D}'', \xi \mathbf{B}^{(2,0)}) \tag{3.11}$$

with $\xi\bar{\xi} = 1$. However the above is not a symmetry of the action functional due to the holomorphic potential term (3.8);

$$S = \frac{s_+ s_-}{4\pi^2} \int_M \text{Tr} \left(\mathbf{B}^{(2,0)} \wedge * \mathbf{F}^{(0,2)} \right) + \dots \tag{3.12}$$

The above situation is exactly same as for Vafa-Witten theory on a Kähler surface. We can use the S^1 symmetry (3.11) to break $N_c = (2, 2)$ supersymmetry down to $N_c = (2, 0)$ supersymmetry by breaking the supersymmetries generated by s_- and \bar{s}_- . We expand (3.12) by one step to get

$$S = \frac{s_+}{8\pi^2} \int_M \text{Tr} \left(i \Upsilon_-^{(2,0)} \wedge * \mathbf{F}^{(0,2)} + \mathbf{B}^{(2,0)} \wedge * \mathbf{D}'' \Psi_-^{(0,1)} \right) + \dots \tag{3.13}$$

Then we see that $(\mathbf{D}'', \mathbf{B}^{(2,0)}, \Psi_-^{(0,1)}) \rightarrow (\mathbf{D}'', \xi \mathbf{B}^{(2,0)}, \bar{\xi} \Psi_-^{(0,1)})$ for $\xi\bar{\xi} = 1$ preserves the action functional. On the one hand the above rotation is not compatible with the supersymmetry generated by s_- since $s_- \mathbf{A}^{(0,1)} = i \Psi_-^{(0,1)}$. On the other hand we can make it compatible with the s_+ supersymmetry by assigning the same $U(1)$ charge to the pair $(\mathbf{B}^{(2,0)}, \Upsilon_+^{(2,0)})$ related by the s_+ supersymmetry, etc.

In the next section we will use the above $S_t^1 \times S_\xi^1$ symmetry to define a $\mathbb{C}^* \times \mathbb{C}^*$ family of $N_c = (2, 0)$ theories. The idea is that all the theories, both the original and the generalized Donaldson-Witten and Vafa-Witten theories, we discussed so far should be viewed as different semi-classical limits governed by different massless degrees of freedom of the same underlying theory.

3.2 A Family of $N_c = (2, 2)$ Models

We begin with generalizing our $N_c = (2, 2)$ model to a \mathbb{C}^* family of $N_c = (2, 2)$ models using the S_t^1 symmetry, whose action is given as (3.4) and (3.5). For that purpose we extend our $N_c = (2, 2)$ supersymmetry by "gauging" the S_t^1 symmetry;

$$\begin{aligned} \{s_+, \bar{s}_+\} &= -i\phi_{++}^a \mathcal{L}_a, \\ \{s_\pm, s_\pm\} &= 0, \quad \{s_+, \bar{s}_-\} = -i\sigma^a \mathcal{L}_a - im \mathcal{L}_{S_t^1}, \quad \{s_+, s_-\} = 0, \\ \{\bar{s}_\pm, \bar{s}_\pm\} &= 0, \quad \{s_-, \bar{s}_+\} = -i\bar{\sigma}^a \mathcal{L}_a - i\bar{m} \mathcal{L}_{S_t^1}, \quad \{\bar{s}_+, \bar{s}_-\} = 0, \\ \{s_-, \bar{s}_-\} &= -i\phi_{--}^a \mathcal{L}_a, \end{aligned} \tag{3.14}$$

where $\mathcal{L}_{S_t^1}$ denotes the Lie derivative defined by the vector field generating the S_t^1 symmetry. Equivalently it is an infinitesimal $U(1)$ gauge transformation for fields with non-vanishing $U(1)$ charge as defined by (3.4) and (3.5). For convenience we write down explicit transformations for bosonic fields

$$\begin{aligned} \mathbf{A}^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & \mathbf{A}^{(1,0)} &:= A^{1,0} + t^{-1}\varphi^{0,1}, \\ \mathbf{B}^{(2,0)} &:= B^{2,0} + t^{-1}B^{1,1} + t^{-2}C^{0,2}, & \mathbf{B}^{(0,2)} &:= B^{0,2} + t\bar{B}^{1,1} + t^2C^{2,0}. \end{aligned} \tag{3.15}$$

The new action functional $S(m, \bar{m})$ is defined by the same formula as given by (3.6) but with modified supersymmetry transformation laws for charged fields under the S_t^1 . We write down the relevant terms depending on the bare mass

$$S(m, \bar{m}) = S + \frac{m\bar{m}}{4\pi^2} \int_M \text{Tr} \left(\varphi^{1,0} \wedge * \varphi^{0,1} + B^{1,1} \wedge * \bar{B}^{1,1} + 4C^{0,2} \wedge * C^{2,0} \right) + \dots \tag{3.16}$$

where the unwritten terms are supersymmetric completions including the bare mass terms of $N_c = (2, 2)$ superpartners of bosonic fields charged under S_t^1 . We remark that the above action functional preserves all the symmetry of the original model. We note that the bare mass terms written above are exactly the Hamiltonian of the S_t^1 action on the space of all bosonic fields. There are two ways of examining the above action functional. One may take the $|m| \rightarrow \infty$ limit. Then the dominant contributions to the path integral come from the critical points of the Hamiltonian of the S_t^1 action. Such critical points are identical to the fixed points of S_t^1 action, equivalently the \mathbb{C}^* action. However this viewpoint is rather limited, as it mainly concerns the moduli space defined by the equations in (3.9). We should not forget that such a moduli space is only a subsystem, and usually does not form a closed system.

A better viewpoint is to rely on the Higgs mechanism. We again take the limit that the bare mass is arbitrarily large. Then we can integrate out everything except for massless degrees of freedom. Here the adjoint scalar fields (Higgs fields) σ and $\bar{\sigma}$ play a crucial role since the effective mass of a field is the sum of the bare mass and the contribution from the expectation values of Higgs scalars. This phenomena can be most directly seen from the anti-commutation relations of supercharges (3.14). Since we have global supersymmetry the expectation values of supersymmetric observables, $\langle \dots \rangle = Z$ in our case, are localized to an integral over the fixed point locus of unbroken global supersymmetry. Consequently the path integral is localized to the kernel of the right hand sides of (3.14) acting on the fields. Then we immediately get the following set of relevant equations for $\mathbf{A}^{0,1}$ and $\mathbf{B}^{2,0}$,

$$\begin{aligned} [\sigma, \varphi^{1,0}] + m\varphi^{1,0} &= 0, \\ [\sigma, B^{1,1}] - mB^{1,1} &= 0, \\ [\sigma, C^{0,2}] - 2mC^{0,2} &= 0, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \bar{\partial}_A \sigma &= 0, \\ [\sigma, \bar{\sigma}] &= 0, \\ [\sigma, B^{2,0}] &= 0, \end{aligned} \tag{3.18}$$

We will now study several limits of these equations.

3.2.1 Three Different Limits

We consider an $SU(2)$ bundle $E \rightarrow M$ for simplicity. The set of equations in (3.17) are the conditions for masslessness of the fields charged under S_t^1 . The second equation in (3.18) implies that σ and $\bar{\sigma}$ can be diagonalized, say $\sigma = \frac{1}{2}diag(a, -a)$. Since $\text{Tr } \sigma^2$ is the gauge invariant object we will consider $a \geq 0$.

We see that there are three (semi-classical) limits governed by different massless degree of freedom while preserving $N_c = (2, 2)$ supersymmetry.

1. Vafa-Witten or a twisted $N = 4$ super-Yang-Mills theory. (i) the gauge symmetry is unbroken $a = 0$. (ii) the gauge symmetry is broken to $U(1)$ $a > 0$ and $a \neq m, 2m$. Then $\varphi^{1,0} = B^{1,1} = C^{0,2} = 0$ is the only solution of (3.17). Equivalently those fields and their $N_c = (2, 2)$ superpartners are all infinitely massive.

2. The gauge symmetry is broken to $U(1)$ and $a = m$ and we have the reduction $E = L \oplus L^{-1}$ Then

$$\begin{aligned}\overline{\partial}_A &= \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & -\overline{\partial}_L \end{pmatrix}, & B^{2,0} &= \begin{pmatrix} \beta^{2,0} & 0 \\ 0 & -\beta^{2,0} \end{pmatrix}, \\ \varphi^{1,0} &= \begin{pmatrix} 0 & 0 \\ \vartheta^{1,0} & 0 \end{pmatrix}, & B^{1,1} &= \begin{pmatrix} 0 & \beta^{1,1} \\ 0 & 0 \end{pmatrix}, \\ & & C^{0,2} &= 0.\end{aligned}\tag{3.19}$$

We have

$$\begin{aligned}F^{0,2} &= 0, \\ i(F - \vartheta^{1,0} \wedge \vartheta^{0,1}) \wedge \omega + \beta^{1,1} \wedge * \overline{\beta}^{1,1} &= 0, \\ \overline{\partial}_L \vartheta^{1,0} &= 0, \\ \overline{\partial}_L \beta^{2,0} + \vartheta^{1,0} \wedge * \beta^{1,1} &= 0, \\ \overline{\partial}_L * \beta^{1,1} &= 0.\end{aligned}\tag{3.20}$$

3. The gauge symmetry is broken to $U(1)$ and $a = 2m$ and we have the reduction $E = L \oplus L^{-1}$ Then

$$\begin{aligned}\overline{\partial}_A &= \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & -\overline{\partial}_L \end{pmatrix}, & B^{2,0} &= \begin{pmatrix} \beta^{2,0} & 0 \\ 0 & -\beta^{2,0} \end{pmatrix}, \\ \varphi^{1,0} &= 0, & B^{1,1} &= 0, \\ C^{0,2} &= \begin{pmatrix} 0 & \gamma^{0,2} \\ 0 & 0 \end{pmatrix}.\end{aligned}\tag{3.21}$$

We have

$$\begin{aligned}F^{0,2} &= 0, \\ \overline{\partial}_L \beta^{2,0} &= 0, \\ iF \wedge \omega + \gamma^{2,0} \wedge \gamma^{0,2} &= 0.\end{aligned}\tag{3.22}$$

3.3 Families of $N_c = (2, 0)$ Models

Following the discussions in Sect. 3.4 and Sect. 4.2 we break the $N_c = (2, 2)$ symmetry down to $N_c = (2, 0)$ supersymmetry generated by s_+ and \bar{s}_+ . The S_ξ^1 -action (3.11) can be extended to all those additional fields introduced for the $N_c = (2, 2)$ model, compared with the original $N_c = (2, 0)$. The S_ξ^1 action is given by

$$\begin{aligned}S_\xi^1 : \left(\mathbf{B}^{(2,0)}, \Upsilon_+^{(2,0)} \right) &\rightarrow \xi \left(\mathbf{B}^{(2,0)}, \Upsilon_+^{(2,0)} \right), \\ S_\xi^1 : \left(\Psi_-^{(0,1)}, \mathbf{H}^{(0,1)} \right) &\rightarrow \bar{\xi} \left(\Psi_-^{(0,1)}, \mathbf{H}^{(0,1)} \right), \\ S_\xi^1 : (\bar{\sigma}, \eta_+) &\rightarrow \bar{\xi}(\bar{\sigma}, \eta_+),\end{aligned}\tag{3.23}$$

and the conjugate fields have the opposite $U(1)_\xi$ -charges. Here we can just follow the procedure in Sect. 4.2 to obtain the general $N_c = (2, 0)$ supersymmetric action functional $S(m, \bar{m}, m_{++}, m_{--})$ is given by

$$S(m, \bar{m}, m_{\pm\pm}) = S(m, \bar{m}) + m_{++}m_{--} \int_M \text{Tr} \left(\mathbf{B}^{(2,0)} \wedge * \mathbf{B}^{(0,2)} + \sigma * \bar{\sigma} \right) + \dots, \quad (3.24)$$

whose new mass terms contain the Hamiltonian of the S_ξ^1 symmetry. The $N_c = (2, 0)$ supercharges s_+ and \bar{s}_+ satisfy the following modified anti-commutation relations

$$s_+^2 = 0, \quad \{s_+, \bar{s}_+\} = -i\phi_{++}^a \mathcal{L}_a - im_{++} \mathcal{L}_{S_\xi^1}, \quad \bar{s}_+^2 = 0. \quad (3.25)$$

Now, in total, we have a $\mathbb{C}^* \times \mathbb{C}^*$ family of $N_c = (2, 0)$ models. From the previous discussions all we need to do is collect all fixed point equations of the supercharges s_+ and \bar{s}_+ . Then the localization equations (3.9) and (3.10) are changed by the following equations

$$\begin{aligned} \mathbf{F}^{(0,2)} &= 0, \\ \mathbf{D}'' * \mathbf{B}^{(2,0)} &= 0, \\ \mathbf{F} \wedge \omega + [\mathbf{B}^{(2,0)}, * \mathbf{B}^{(0,2)}] - \frac{1}{2} [\sigma, \bar{\sigma}] \omega \wedge \omega &= 0, \\ \mathbf{D}' \sigma + m \varphi^{0,1} &= 0, \\ [\sigma, \mathbf{B}^{(2,0)}] - m B^{1,1} - 2m C^{0,2} &= 0, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} [\phi_{++}, \mathbf{B}^{(2,0)}] + m_{++} \mathbf{B}^{2,0} &= 0, \\ [\phi_{++}, \sigma] + m_{++} \sigma &= 0, \\ [\phi_{++}, \bar{\phi}_{--}] &= 0, \\ d_A \phi_{++} &= 0. \end{aligned} \quad (3.27)$$

By sending all the bare masses to infinity we have various semi-classical limits governed by different massless degrees of freedom.

3.3.1 Generalized Seiberg-Witten theory

For our purpose it is suffice to examine a limit $m_{\pm\pm} \rightarrow \infty$ by setting $m = \bar{m} = 0$. For simplicity we consider the $SU(2)$ case. Then we can follow the discussions in Sect. 4.2.2 and see that the path integral can be written as the sum of contributions from two branches;

- branch (i): On a generic point on the vacuum moduli space we have the trivial fixed point $\mathbf{B}^{(0,2)} = 0$ and the fixed point locus is the moduli space \mathcal{N} of stable Higgs bundles,

$$\begin{aligned}\mathbf{F}^{(0,2)} &= 0, \\ \mathbf{F} \wedge \omega &= 0.\end{aligned}\tag{3.28}$$

Hence we recover the generalization Donaldson-Witten theory in Sect. 7.2.

- branch (ii): The $SU(2)$ symmetry is broken down to $U(1)$. We have $E = L \oplus L^{-1}$ and

$$\mathbf{D}'' = \begin{pmatrix} \mathbf{d}_L'' & 0 \\ 0 & \mathbf{d}_L' \end{pmatrix}, \quad \mathbf{B}^{(2,0)} = \begin{pmatrix} 0 & \mathbf{b}^{(2,0)} \\ 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \tag{3.29}$$

where $\mathbf{d}_L'' = \bar{\partial}_L + \vartheta^{1,0}$ and $\mathbf{b}^{(2,0)} = \beta^{2,0} + \beta^{1,1} + \gamma^{0,2}$ takes values in L^{-2} . The fixed point equation are

$$\begin{aligned}\mathbf{F}^{(0,2)} &= 0, \\ \mathbf{d}_L' \alpha &= 0, \\ \mathbf{d}_L'' * \mathbf{b}^{(2,0)} &= 0, \\ i\mathbf{F}_L \wedge \omega - \mathbf{b}^{(2,0)} \wedge * \mathbf{b}^{(0,2)} + \alpha \bar{\alpha} \omega^2 &= 0.\end{aligned}\tag{3.30}$$

The above set of equations is a spectral generalization of Abelian Seiberg-Witten equation. For $\vartheta^{1,0} = \beta^{1,1} = \gamma^{0,2} = 0$ the above equation reduces to the usual Seiberg-Witten equation for a special set of Seiberg-Witten classes [20][39].

It is a well-established fact that Donaldson-Witten (DW) theory is equivalent to Seiberg-Witten (SW) theory [6]. One of the strong evidences, or vice versa, for such equivalence is the S -duality of Vafa-Witten (VW) theory, which has both DW and SW theories as two different semi-classical limits after the massive perturbation. The S -duality, for $SU(2)$ and $SO(3)$, implies that one can recover the entire partition function from one of such semi-classical limits. We expect similar relations between the generalized versions. We believe the moduli space of the generalized Seiberg-Witten equations (3.30) and associated invariants deserve detailed study. It remains to be seen if our invariants contain new information on smooth structures beyond Seiberg-Witten invariants.

Acknowledgements

This paper is based on a chapter of the author's PhD thesis defended by Nov. 1999 in University of Amsterdam. I would like to thank my adviser H. Verlinde as well as R. Dijkgraaf and C. Hofman for valuable discussions. This research is supported by a Pioneer Fund of NWO and DOE Grant # DE-FG02-92ER40699.

References

- [1] E. Witten, *Topological quantum field theory*, Commun. Math. Phys. **117** (1988) 353.
- [2] S.K. Donaldson, *Polynomial invariants for smooth 4-manifolds*, Topology **29** (1990) 257.
- [3] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Clarendon Press, Oxford 1990.
- [4] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B 426** (1994) 19.
- [5] N. Seiberg and E. Witten, *Monopole, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, [hep-th/9408099](#).
- [6] E. Witten, *Monopoles and four manifolds*, Math. Research Lett. **1** (1994) 769. [hep-th/9411102](#).
- [7] S. Hyun, J. Park and J.-S. Park, *Topological QCD*, Nucl. Phys. **B 453** (1995) 119. [hep-th/9503201](#)
- [8] J.M.F. Labastida, M. Mario, Non-Abelian Monopoles on Four-Manifolds Nucl. Phys. **B448** (1995) 373. [hep-th/9504010](#) ;
Polynomial Invariants for SU(2) Monopoles Nucl. Phys. **B456** (1995) 633. [hep-th/9507140](#)
- [9] G. Moore and E. Witten, *Integration over the $u-$ plane in Donaldson theory*, [hep-th/9709193](#).
- [10] A. Losev, N. Nekrasov and S. Shatashvili, *Issues in topological gauge theory*, Nucl. Phys. **B534** (1998) 549. [hep-th/9711108](#)

- [11] Marcos Marino, Gregory Moore, Integrating over the Coulomb branch in $N=2$ gauge theory, *Nucl. Phys. Proc. Suppl.* **68** (1998) 336. [hep-th/9712062](#)
- [12] Marcos Marino, Gregory Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, *Commun. Math. Phys.* **199** (1998) 25. [hep-th/9802185](#),
- [13] Marcos Marino, Gregory Moore, Donaldson invariants for nonsimply connected manifolds [hep-th/9804104](#)
- [14] C. H. Taubes, *$SW \rightarrow GR$. From the Seiberg-Witten equations to pseudo-holomorphic curves*, *Bull. Am. Math. Soc.* **9** (1996) 845.
- [15] S. Wang, *A vanishing theorem for Seiberg-Witten invariants*, *Math. Res. Letters* **2** (1995) 305.
- [16] R. Fintushel and R.J. Stern, *Nondiffeomorphic symplectic 4-manifolds with the same Seiberg-Witten invariants*, [math.SG/9811019](#).
- [17] C.T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, *J. Amer. Math. Soc.* **1** (1988) 867.
- [18] C.T. Simpson, *Higgs bundles and local systems*, *Publ. Math. I.H.E.S.* **75** (1992) 5.
- [19] C.T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, I:*Publ. Math. I.H.E.S.* **79** (1994) 47; II:*Publ. Math. I.H.E.S.* **80** (1994) 5.
- [20] C. Vafa and E. Witten, *A strong coupling test of S-Duality*, *Nucl. Phys. B* **431** (1994) 3, [hep-th/9408074](#).
- [21] J. Yamron, *Topological actions in twisted supersymmetric theories*, *Phys. Lett. B* **213** (1988) 353.
- [22] J.-S. Park, *Cohomological field theories with Kähler structure*, *Adv. Theo. Math. Phys.* in press, [hep-th/9910209](#)
- [23] J.-S. Park, *$N=2$ topological Yang-Mills theory on compact Kähler surfaces*, *Comm. Math. Phys.* **163** (1994) 113, [hep-th/9304060](#) ;
Holomorphic Yang-Mills theory on compact Kähler manifolds, *Nucl. Phys. B* **423** (1994) 559, [hep-th/9305095](#).

- [24] E. Witten, *Supersymmetric Yang-Mills theory on a four-manifold*, J. Math. Phys. **35** (1994) 5101, [hep-th/9303195](#).
- [25] S. Hyun and J.-S. Park, *N=2 topological Yang-Mills theories and Donaldson's polynomials*, J. Geom. Phys. **20** (1996) 31, [hep-th/9404009](#).
- [26] S.K. Donaldson, *Anti-self-dual Yang-Mills connections on complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **3** (1985) 1;
Infinite determinants, stable bundles and curvature, Duke Math. J. **54** (1987) 231.
- [27] K.K. Uhlenbeck and S.T. Yau, *The existence of Hermitian Yang-Mills connections on stable bundles over Kähler manifolds*.
- [28] D. Kaledin, *Hyperkähler structures on total spaces of holomorphic cotangent bundles*, [alg-geom/9710026](#).
- [29] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **3**, **55** (1987) 59.
- [30] A. Fujiki, *HyperKähler structure on the moduli space of flat bundles*, in "Prospects in complex geometry", Lecture Notes in Math. p. 1-83, Springer-Verlag, 1991.
- [31] R. Donagi and E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, [alg-geom/9507017](#).
- [32] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is a T-duality*, Nucl. Phys. **B** **479** (1996) 243, [hep-th/9612121](#).
- [33] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, [hep-th/9804131](#).
- [34] S. Cecotti and C. Vafa, *Topological anti-topological fusion*, Nucl. Phys. **B367** (1991) 359.
- [35] E. Witten *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159, [hep-th/9301042](#).
- [36] C. Hofman and J.-S. Park, *Sigma Models for bundles on Calabi-Yau: a proposal for matrix string compactifications*, Nucl. Phys. B561 (1999) 125, [hep-th/9904150](#);

Cohomological Yang-Mills Theories on Kahler 3-Folds, Nucl. Phys. B in press, [hep-th/001010](#)

- [37] E. Witten, *Introduction to cohomological field theories*, Int. J. Mod. Phys. **A6** (1991) 2775.
- [38] R. Dijkgraaf and G. Moore, *Balanced topological field theories*, Commun. Math. Phys. **185** (1997) 411, [hep-th/9608169](#).
- [39] R. Dijkgraaf, J.-S. Park and B.J. Schroers, *$N=4$ supersymmetric Yang-Mills theory on a Kähler surface*, [hep-th/9801066](#).
- [40] J.-S. Park, *Monads and D-instantons* Nucl. Phys. **B 493** (1997) 198, [hep-th/9612096](#).
- [41] J. M. F. Labastida and C. Lozano, *Mass perturbations in twisted $N=4$ supersymmetric gauge theories*, Nucl. Phys. **B518** (1998) 37, [hep-th/9711132](#).
- [42] J. M. F. Labastida and C. Lozano, *Duality in twisted $N=4$ supersymmetric gauge theories in four dimensions*, Nucl. Phys. **B537** (1999) 203, [hep-th/9806032](#).